HIGH QUARTAN FACTORISATIONS AND PRIMES.

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1. *Quartans, Octavans.* THE numbers discussed in this Paper are of forms

$$N=x^4+y^4, \text{ and } N=x^8+y^8 \ldots\ldots\ldots\ldots\ldots(1).$$

For shortness' sake these algebraic forms will be styled *Quartans*, and *Octavans*, respectively.

1*a*. *Working condition.* For sake of brevity the *working condition* of "x prime to y" is generally assumed. With this condition N or $\frac{1}{2}N$ is always odd; in fact

x, y one odd, one even, give N *odd*.....(1*a*),

x, y both odd give N even, and $\frac{1}{2}N$ odd..... .....(1*b*).

In the latter case $\frac{1}{2}N$ will be styled a *Half-Quartan** or *Half-Octavan*.*

2. *Notation.* All symbols denote *integers*;

ω, Ω denote *odd* numbers; ε, e denotes *even* nnmbers; i *any* integer; p denotes a *prime*; N is defined in (1).

3. *Linear Forms.* The following simple linear forms of N and $\frac{1}{2}N$ to various moduli are to be noted as occurring under the "conditions" named:

Condition.	*Quartans.*	*Half-Quartans.*	*Octavans.*	*Half-Octavans.*
None	$16n+1$	$8n+1$	$32n+1$	$16n+1$.....(2*a*),
x or $y=3i$	$16.3n+1$	$8.3n+1$	$32.3n+1$	$16.3n+1$.....(2*b*),
x or $y=5i$	$16.5n+1$	$8.5n+1$	$32.5n+1$	$16.5n+1$.....(2*c*),
x or $y=5i$	$100n+\varepsilon,1$†	$100n+\omega,3$†	$100n+\varepsilon,1$†	$100n+\omega,3$†...(2*d*),
x or $y\neq 5i$	$100n+\omega,7$†	$100n+\varepsilon,1$†	$100n+\omega,7$†	$100n+\varepsilon,1$†....(2*e*),

$$x=\omega_1\omega_2,\ m(2\omega_1)^4+\omega_1^4+y^4,\ m.2^2\omega_1^4+\frac{\omega_1^4+y^4}{2},\ m.2^5\omega_1^8+\omega_1^8+y^8,\ m.2^4\omega_1^8+\frac{\omega_1^8+y^8}{2}\ldots(2f).$$

* It is proposed to reserve the terms *Semi-Quartan*, and *Semi-Octavan* for the forms $N=x^2+y^4$, $N=x^4+y^8$, respectively.

† Here ε, 1; ω, 3; ω, 7 are to be read *arithmetically* as (the tens and units) digits of N and $\frac{1}{2}N$.

4. *Quadratic Forms* (of Quartans). Every Quartan (N) and Half-Quartan ($\frac{1}{2}N$) may be expressed (algebraically) in the following four quadratic forms

$$N = a^2 + b^2 = (x^2)^2 + (y^2)^2 \quad \ldots\ldots (3a),$$

$$= c^2 + 2d^2 = (x^2 - y^2)^2 + 2(xy)^2 \quad \ldots\ldots (3b),$$

$$= e^2 - 2f^2 = (x^2 + y^2)^2 - 2(xy)^2 \quad \ldots\ldots (3c),$$

$$= 2f'^2 - e'^2 = 2(x^2 \mp xy + y^2)^2 - (x^2 \mp 2xy + y^2)^2 \ldots (3d),$$

$$\tfrac{1}{2}N = a^2 + b^2 = \left(\frac{x^2 + y^2}{2}\right)^2 + \left(\frac{x^2 - y^2}{2}\right)^2 \quad \ldots\ldots (4a),$$

$$= c^2 + 2d^2 = (xy)^2 + 2\left(\frac{x^2 - y^2}{2}\right)^2 \quad \ldots\ldots (4b),$$

$$= e^2 - 2f^2 = (x^2 \mp xy + y^2)^2 - 2\left(\frac{x^2 \mp 2xy + y^2}{2}\right)^2 \ldots (4c),$$

$$= 2f'^2 - e'^2 = 2\left(\frac{x^2 + y^2}{2}\right)^2 - (xy)^2 \quad \ldots\ldots (4d).$$

These will be styled—for shortness—the (a, b), (c, d), (e, f), (e′, f′) partitions. Note that—disregarding signs—

$$\text{In } N;\ 2a \text{ or } 2b = c + e,\ d = f = f' - e' \quad \ldots\ldots (5a),$$

$$\text{In } \tfrac{1}{2}N;\ a \text{ or } b = d,\ b \text{ or } a = f',\ c = e' \quad \ldots\ldots (5b).$$

When N or $\frac{1}{2}N$ is odd, the "terms" of each partition are one odd, and one even; it is usual to take

$$a, c, e, e' \textit{ odd};\ b, d, f, f' \textit{ even} \quad \ldots\ldots (6).$$

The (e, f), (e′, f′) partitions, having the same determinant ($D = +2$), are not to be considered *different forms*, being merely different expressions of the *same form*; they are in fact immediately interconvertible by the relations

$$e' = e \mp 2f,\ f' = e \mp f;\ e = 2f' \mp e',\ f = f' \mp e' \quad \ldots\ldots (7).$$

The (a, b), (c, d), (e, f) forms, having different determinants ($D = -1, -2, +2$) are to be considered *different forms*; they are however not independent, but are so connected that any one of them may be derived* from the other two.

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* See Art. 24 to 36 of the author's Paper on *Connexion of Quadratic Forms* in *Proc. Lond. Math. Soc.*, Vol. XXVIII, 1896.

When N or $\frac{1}{2}N$ is prime, each of the (a, b), (c, d) forms is *unique*: the (e, f), (e′, f′) forms may also be considered *unique*; for, although N and $\frac{1}{2}N$ may be expressed in an infinite number of ways in these forms, yet all these expressions are mere *automorphs* of the (e, f), (f′, e′) forms given above, which are in fact their *Base-forms*, *i.e.* the forms with minimum values (when the upper signs are used) of e, f, e′, f′; the Base-forms are marked by the property—

$$e > 2f,\ f' > e' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a).$$

When N or $\frac{1}{2}N$ is composite, and contains different primes as factors (so as not to be merely a power of a single prime), it is always expressible—but not as a rule algebraically (unless the partitions of the component factors be known)—in more than one way in each of the above 2^{ic} forms. Each resolution into a pair of (unequal) factors gives rise to a different base-form in the (e, f) and (f′, e′) partitions; but it is beyond the scope of this Paper to enter into these merely arithmetical ways of 2^{ic} partition.

4*a*. *Expression of given* 2^{ic} *forms as Quartans, &c.* When an *odd* number N or $\frac{1}{2}N$ is given in some *one* of the 2^{ic} forms (a, b), (c, d), (e, f), (e′ f′), it will be expressible as a Quartan or Half-Quartan with elements (x, y) as below, provided the quantities under the radicals are *perfect squares*; (this involves *two conditions* in each case):—

$$N = a^2 + b^2\ ;\quad x = \sqrt{a},\quad y = \sqrt{b} \quad \ldots\ldots\ldots\ldots (3a'),$$

$$N = c^2 + 2d^2;\quad x \text{ or } y = [\tfrac{1}{2}(\sqrt{c^2 + 4d^2} \pm c)]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (3b'),$$

$$N = e^2 - 2f^2;\quad x \text{ or } y = [\tfrac{1}{2}(\sqrt{e^2 - 4f^2} \pm e)]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (3c'),$$

$$N = 2f'^2 - e'^2;\quad x \text{ or } y = \tfrac{1}{2}[\pm\sqrt{e'} \pm \sqrt{4f' - 3e'}] \quad \ldots\ldots\ldots\ldots (3d'),$$

$$\tfrac{1}{2}N = a^2 + b^2\ ;\quad x = (a \pm b)^{\frac{1}{2}},\ y = (a \mp b)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (4a'),$$

$$\tfrac{1}{2}N = c^2 + 2d^2;\quad x \text{ or } y = [\sqrt{c^2 + d^2} \pm d]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (4b'),$$

$$\tfrac{1}{2}N = e^2 - 2f^2;\quad x \text{ or } y = \tfrac{1}{2}[\pm\sqrt{2f} \pm \sqrt{4e - 6f}) \quad \ldots\ldots\ldots\ldots (4c'),$$

$$\tfrac{1}{2}N = 2f'^2 - e'^2;\quad x \text{ or } y = [\sqrt{f'^2 - e'^2} \pm f']^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (4d').$$

Note that the forms (e, f), (e′, f′) to be used in Results (3*c*′), (3*d*′), (4*c*′), (4*d*′) must be their *Base-forms*, (see the requisite condition (7*a*)).

4*b*. *Quadratic Forms* (of Octavans). Octavans and Half-Octavans may be expressed (algebraically) in the same 2^{ic}

forms as the Quartans and Half-Quartans of which they are merely specialised forms. All the results of Art. 4 are applicable to them also, except that x^2, y^2 must be substituted for the x, y of those formulæ.

5. *Divisors.* When x, y are both odd, then N is *even*, and has the divisor 2 (but not 4), so that $\frac{1}{2}N$ is always *odd* (cf. Art. 1*a*). All other divisors are either primes of the forms

$$p = 8w+1 \text{ for Quartans}; \quad p = 16w+1 \text{ for Octavans} \ldots\ldots\ldots\ldots (8);$$

or powers of such primes, or products of such primes and their powers.

5*a*. 2^{1c} *forms of divisors.* Every odd divisor (Q) of N or $\frac{1}{2}N$ is expressible—but not, as a rule, algebraically—in *each* of the above 2^{1c} forms of Art. 4,

$$Q = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 = 2f'^2 - e'^2 \ldots\ldots\ldots (8a),$$

and these partitions are all *unique* (in the sense above explained, Art. 4) in the case of prime divisors.

5*b*. 2^{1c} *forms of large divisors.* In the case of composite Quartans and Octavans, and their halves, say N or $\frac{1}{2}N = q.Q$, the (a, b), (c, d), (e, f), (f′, e′) forms of either factor (say Q) can always be formed by the process of *conformal* division* when the corresponding 2^{1c} forms of the other factor (q) are known, provided the latter factor be either a *prime*, or a power of a prime (and can sometimes also be found by that process when q is composite).

When N is large, and has one prime, or power of a prime, divisor ($q = p$ or p^κ) so small that its 2^{1c} parts (a_1, b_1, &c.) are either known, or can be easily found, this process affords an *easy way* of finding the 2^{1c} parts of the other factor (Q) *even when very large*: this is important, as the direct determination of the 2^{1c} partitions of a very large number (as Q) is usually pretty laborious.

Ex. Take $N = 2^{64}+1 = q.Q$; where $q = 274177$.

The 2^{1c} partitions of the small factor q are easily found (by trial). Hence by (3*a* to *d*),

$$Q = \frac{N}{q} = \frac{1^2+(2^{32})^2}{89^2+516^2} = \frac{(2^{32}-1)^2+2.(2^{16})^2}{623^2+2.28^2} = \frac{(2^{32}+1)^2-2.(2^{16})^2}{575^2-2.148^2}.$$

* *Conformal Division* is division with preservation of 2^{1c} form. For the details of this process, and for the conditions for carrying it out successfully when q is composite, see the author's Paper on *Connexion of Quadratic Forms* above quoted, Art. 15 to 23.

The process of conformal division* now gives *directly* the required partitions of the large factor $Q = 67280421310721$, viz.

$$Q = (8083111^2 + 1394180^2) = (8192757^2 + 2.282094^2) = 9007423^2 - 2.2631848^2,$$

and the last one gives also $Q = (2.6375575^2 - 3743727^2)$. These results would be difficult to obtain directly.

6. *Factorisation-Tables.* The author has had 8 Tables of the factorisation of these numbers compiled, as below (Art. 6*a*, *b*):—

6*a*. *General Tables.* The factorisation of *all* such numbers (x, y, both varying) has been worked out *completely*†‡ up to the limits named in the Abstract below, which shows also the total number (n) of each class factorised.

	N or $\frac{1}{2}N$ =	(x^4+y^4)	$\frac{1}{2}(x^4+y^4)$	x^5+y^5	$\frac{1}{2}(x^5+y^5)$
Limits of	x =	$\infty > 53$	$\infty > 65$	$\infty > 11$	$\infty > 11$
	y =	$\infty > 54$	$\infty > 55$	$\infty > 10$	$\infty > 9$
	N or $\frac{1}{2}N >$	9 million	9 million	215 million	107 million
	Number (n)	534	367	26	11

The Quartan factorisation was done by the large Factor-Tables, and therefore extends only to their limit (9 million). The Octavans being few in number, the factorisation was carried much further by special means.

6*b*. *Simple* 4*-tan and* 8*-van Tables* ($x = 1$). The factorisation of all *simple* Quartans and Octavans, $N = (1 + y^4)$, and $(1 + y^8)$, has been carried out§‖ as completely as possible with the means available (to be described in Art. 8 to 15) up to the high limits named in the Abstract below, which shows also the degree of success attained, *i.e.* the number of each class completely or partially factorised.

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* See footnote * above.

† The 4-tan Tables by Miss E. Cooper, and checked by Miss A. Woodward, under the author's superintendence. The 8-van Tables by the author himself.

‡ These Tables are not published: the 4-tan Tables could be readily prepared by any computer (from the large Factor-Tables).

§ These 4-tan Tables were computed by the late Mr. C. E. Bickmore and the present author jointly, as far as $y = 100$. The 4-tan Tables beyond $y = 100$, and the 8-van Table, were computed partly by the author himself, partly by two Assistants (Miss A. Cole, and Miss E. Cooper), and checked throughout by one of them and in part also by another Assistant (Miss A. Woodward), under the author's superintendence.

‖ These Tables are thought too extensive for publication herewith: it is hoped to publish them in a separate Work.

	Complete Factoris'n.	*Factorisation.*	*Limit.*
N or $\frac{1}{2}N$; y	*Range of y; Number*	*Range of y ; Compl. Part'l. Fail*	*Total. N* or $\frac{1}{2}N$
$(1+y^4)$; ϵ	2 to 226; 113 (All)	228 to 1000; 219, 63; 105	500; 10^{12}
$\frac{1}{2}(1+y^4)$; ω	1 to 265; 133 (All)	267 to 999; 230, 35; 102	500; $\frac{1}{2}10^{12}$
$(1+y^8)$; ϵ	2 to 18; 9 (All)	20 to 160; 16, 42; 13	80; 4.10^{17}
$\frac{1}{2}(1+y^8)$; ω	1 to 19; 10 (All)	21 to 159; 16, 38; 16	80; 2.10^{17}

7. *Quartan, &c., Octavan, &c., Primes, and Factors;* (Tab. I. to VII.). These Tables give *complete lists* of all Quartan, Half-Quartan, Octavan, and Half-Octavan primes, and also of all High Prime Factors ($p > 9$ million) of simple Quartans, Half-Quartans, Octavans, and Half-Octavans, up to the high limits named in the Abstract below, and also a few beyond those limits: the Abstract shows also the *number* of primes of each class within the limits stated; those beyond the limits (of x, y or p) stated are numbered as "Extra."

The Quartan results are classified in six Tables (Tab. I.—VI.) as in the Abstract below: the Octavan results are similarly classified, but (being few in number) are all printed in one Table (Tab. VII.). Inasmuch as Octavans are merely specialised Quartans, some few of the Octavan results appear in Tab. I.—VI. (when comprised within the limits of these Tables) as well as in Tab. VII.

	Tab.	p	*Limits of p*	x	y	*Numb.*	*Extra*
Quartans & Half-Quartans	I.	(x^4+y^4)	$\not> 9$ million	$\omega \not> 53$	$\epsilon \not> 54$	232	.
	II.	$\frac{1}{2}(x^4+y^4)$	$\not> 9$ million	$\omega \not> 65$	$\omega \not> 55$	166	.
	III.	$(1+y^4)$	$> 9.10^6, \not> 27.10^8$	1	$\epsilon > 54, \not> 226$	20	2
	IV.	$\frac{1}{2}(1+y^4)$	$> 9.10^6, \not> 25.10^8$	1	$\omega > 65, \not> 265$	18	.
Factors	V.	$\frac{1}{\mu}.(1+y^4)$	$> 9.10^6, \not> 10^9$	1	$\epsilon > 110, \not> 1000$	108	5
	VI.	$\frac{1}{\mu}.\frac{1}{2}(1+y^4)$	$> 9.10^6, \not> 10^9$	1	$\omega > 131, \not> 999$	102	4
Octavans & Half-Octavans	VII.	(x^8+y^8)	$\not> 9$ million	$\omega \not> 7$	$\epsilon \not> 6$	3	.
	,,	$\frac{1}{2}(x^8+y^8)$	$\not> 9$ million	$\omega \not> 7$	$\omega \not> 7$	2	.
	,,	(x^8+y^8)	$> 9.10^6, \not> 43.10^{11}$	1	$\epsilon > 6, \not> 36$	.	1
	,,	$\frac{1}{2}(x^8+y^8)$	$> 9.10^6, \not> 18.10^9$	1	$\omega > 7, \not> 19$	2	1
Factors	,,	$\frac{1}{\mu}.(1+y^8)$	$> 9.10^6, \not> 10^9$	1	$\epsilon > 10, \not> 160$	9	.
	,,	$\frac{1}{\mu}.\frac{1}{2}(1+y^8)$	$> 9.10^6, \not> 10^9$	1	$\omega > 11, \not> 159$	8	.

Divisor μ. Note that in Tab. V., VI., VII., the entry μ (in the column headed μ) indicates that the divisor (μ) needed is > 100.

Nearly all the primes here reported were detected in the course of the factorisation described in Art. 6*a*, *b*: the magni-

tude of the High Primes is therefore generally limited by the extent of those Tables, and by the means of factorisation available (Art. 8), *i.e.*

Factorisation-Limit.	*High Prime-Limit.*
In 4-tans, $y \gtrless 1000$; In 8-vans, $y \gtrless 160$	Usual, $p \gtrless 105.10^{7}$; Special, $p \gtrless 26.10^{8}$.

The few marked "Extra" in the Abstract above were either specially worked out by, or (in a few cases only) obtained from other Works.

7*a*. *High Primes.* An Abstract of the magnitudes of the High Primes here reported is given below. Most of them are believed to be *new* (*i.e.* not previously published): those previously published—so far as known to the author—are marked in the Tables (Tab. III. to VII.) by a capital letter (B, D, &c.) which serves to indicate the name of the discoverer and Work in which published according to the scheme in the footnote.*

p	7-fig.	8-fig.	9-fig.	10-fig.	Total
x^4+y^4	1,	5,	11,	5;	22
$\frac{1}{2}(x^4+y^4)$	.,	7,	9,	2;	18
$\frac{1}{\mu}(x^4+y^4)$	2,	69,	42,	.;	113
$\frac{1}{\mu}.\frac{1}{2}(x^4+y^4)$	6,	65,	34,	1;	106
Total	9,	146,	96,	8;	259

p	7-fig.	8-fig.	9-fig.	10-fig.	Total
x^8+y^8	.	.,	1,	.;	1
$\frac{1}{2}(x^8+y^8)$	.	1,	2,	.;	3
$\frac{1}{\mu}(x^8+y^8)$	.	3,	6,	.;	9
$\frac{1}{\mu}.\frac{1}{2}(x^8+y^8)$	.	4,	4,	.;	8
Total	.	8,	13,	.;	21

7*b*. *Few prime binomials of high order.* The number of primes which are simple binomials, or half-binomials of even order, *i.e* of form

$$p=(1+y^e), \text{ or } p=\tfrac{1}{2}(1+y^e), \text{ where } e=2, 4, 8, \&c.,$$

will be found to decrease rapidly as e increases: the comparison may be made in two ways, viz.

(1) within same range of y; (2) within same range of p.

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* As far as known to the author only 15 of these High Primes had been previously published, or, previously discovered by others, viz.:

2 marked D, due to E. Desmarest, see *Théorie des Nombres*, Paris, 1852, p. 286.
1 marked Lf, due to W. Looff, see *Nouv. Ann. de Mathém*, 2° Sér. t. xiv., 1885, p. 116.
1 marked Da, due to W. B. Davis, see Liouv. *Journ. de Mathém. pures et appl.*, Sér. 2°, t. xi, 1866, pp. 188—190.
4 marked L, due to F. Landry, see *Décomposition des Nombres* $(2^n \pm 1)$ &c., Paris, 1869.
1 marked C, due to the author, see *Quarterly Journ. of P. & Appl. Maths.*, v. 35, 1903, p. 21.
6 marked B, due to the late C. E. Bickmore and the author jointly (not published).
1 marked J., (among the last six) confirmed by Morgan Jenkins in letter to the author.

The following Table gives the data*, *i.e.* the *number* of primes (n) of each order for same ranges of y and p; [the numbers when $e=1$ have been added for sake of comparison].

Form $p=(1+y^4)$		*Form* $p=\frac{1}{2}(1+y^4)$	
Limit of y or p; $e=$	1, 2, 4, 8	Limit of y or p; $e=$	1, 2, 4, 8
$y>36$; $n=$	13, 10, 8, 2	$y>19$; $n=$	5, 7, 7, 3
$y>226$; $n=$	40, 37, 31, ?	$y>265$; $n=$	33, 44, 33, ?
$y>15000$; $n=$	1755, 1199, ?, ?	$y>14999$; $n=$	951, 1288, ?, ?
$p>9$ million; $n=602568$†,	302, 11, 2	$p>9$ million; $n=602568$†,	445, 15, 2

8. *Congruence-Tables.* The factorisation of $N=(1+y^4)$ and $(1+y^8)$ up to the high limits tried (Art. 6*b*), and the detection of the High Primes reported (Art. 7*a*) were rendered possible chiefly by the preparation of extensive Tables‡§ of solutions of the two Congruences

$$y^4+1\equiv 0 \pmod{p \text{ and } p^2}, \quad \text{and } y^8+1\equiv 0 \pmod{p \text{ and } p^2} \ldots\ldots (9).$$

These Tables are now complete and continuous up to the following high limits

For y^4+1, up to $p=32441$, $p^2=193^2$; For y^8+1, up to $p=9857$, $p^2=97^2$,

and have been worked out also for *many* higher primes; but in this latter part the Tables are not continuous.

9. *Reduction of Fractions.* Many of the Congruence-solutions following are presented in the *form of fractions*, thus

$$y\equiv \frac{N}{D}, \pmod{M} \ldots\ldots (10),$$

where N, D, M denote *numerator*, *denominator*, and *modulus*, respectively.

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* The number (n) of primes $p=(1+y^2)$ and $\frac{1}{2}(1+y^2)$ were obtained from MS. Factorisation Tables of these forms extending to $y=15000$, compiled by the author: these are nearly ready for publication.

† This number, the total number of primes <9 million is taken from Glaisher's *Factor-Table* for the sixth million, 1883, page 32: it is only approximate, certain errors having been found in some of the large Factor-Tables since the count was made.

‡ These Tables were prepared in part by the author himself; but for the most past by an Assistant (Miss E. Cooper) under the author's superintendence. One of the Tests described in Art. 13 was always applied; and the additions described in Result (26) were always checked by the author himself and by another Assistant (usually Miss C. Woodward).

§ These Congruence-Tables are far too extensive for publication herewith: it is hoped to publish them hereafter in a separate Work, along with the large Factorisation-Tables described in Art. 6*b*.

Hence,

$$y \equiv \frac{N \pm m.M}{D}, \pmod{M} \ldots\ldots\ldots (10a).$$

To reduce this to an integer, it is only necessary to determine m so that the numerator $(N \pm mM)$ shall be divisible by D: this gives the required *integral solution* (y). This is easy when D is small, but increases in difficulty as D increases: hence, when D is *composite*, it is often convenient to resolve it into its factors, say $D = D_1.D_2 \ldots D_r$, and to reduce each factor separately by the above process; Thus

(1) Reduce N/D_1 to an integer, say N_1 (as above).

(2) Reduce N_1/D_2 to an integer, say N_2 (as above), and so on.

10. *Construction of Congruence-Tables.* This consists of two distinct steps for each modulus (p or p^κ).

Step i. Finding one root (y) of the Congruences, Art 11 to 11*f*.

Step ii. Finding the remaining roots (y', y'', &c., each $< p$) from a known root (y), Art. 12.

11. Step i. *Finding one root* (y). This may be done in a variety of ways. Several of these will be described in the following Articles (11*a* to *f*); each has its own special conveniences, as will be explained below.

1°. From a known factorisation $N = (X^4 + Y^4)$ or $(X^8 + Y^8)$.....Art. 11*a*.

2°. From a known power-congruence $a^{4\xi'}$ or $a^{8\xi''} \equiv -1$.....Art. 11*b*.

3°. From *two* known 2^{ic} partitions, either (a, b), (c, d), (e, f).....Art. 11*c*.

4°. From a known factorisation $N = (\alpha X^2)^4 + (\beta Y^2)^4$
with a known 2^{ic} partition $(at^2 \pm \beta u^2)$, or $(t^2 \pm a\beta u^2)$.....Art 11*d*.

Methods 1°, 2° are general methods, applicable (with suitable change of the index n) to *any* binomial congruences $y^n \pm 1 \equiv 0 \pmod{p \text{ or } p^\kappa}$. Method 3° is a special method for quartans, and method 4° is a special method for octavans.

11*a*. Method 1°. *From a known factorisation.* If there be given

$$N = (X^4 + Y^4) \text{ or } = (X^8 + Y^8) \equiv 0 \pmod{p \text{ or } p^\kappa} \ldots (11),$$

this gives at once *four* roots of either congruence in the fractional form (Art. 9)

$$y \equiv \pm X/Y, \text{ or } y \equiv \pm Y/X \pmod{p \text{ or } p^\kappa} \ldots\ldots (11a),$$

for every prime (p) and prime-power (p^κ) contained in N.

This Method is very convenient, as the reduction of the fraction is easy (Art. 9), one of the terms (X, Y) being usually small: unfortunately it is limited by the powers of factorisation (which in the case of Octavans are very small). Of course, if $X=1$, then two roots $(y = \pm Y)$ are given *at sight.*

Ex. Given $31^4+28^4=17.90481$; to solve $y^4+1\equiv 0 \pmod{p=90481}$,

Here $\pm y \equiv \frac{31}{28} \equiv \frac{31+3.90481}{28} \equiv \frac{271474}{28} = \frac{19391}{2} \equiv \frac{-71090}{2} = -35545$, are two roots.

11*b*. METHOD 2°. *From a known power-congruence.* If there be given

$$a^{4\xi'} \equiv -1, \text{ or } a^{8\xi''} \equiv -1 \pmod{p \text{ or } p^\kappa} \ldots\ldots (12),$$

then $y =$ least residue of $\pm a^{\xi'}$ or $\pm a^{\xi''} \pmod{p \text{ or } p^\kappa} \ldots (12a)$,

are two roots of $y^4+1 \equiv 0$ or $y^8+1 \equiv 0 \pmod{p \text{ or } p^\kappa}$ respectively.

Thus any primitive root (g) of p or p^k will always suffice to give two roots (y). To save (numerical) labor, it is desirable that ξ' or ξ'' should be as small as possible; the minimum of ξ' or ξ'' is secured when $8\xi'$ or $16\xi''$ respectively $=\xi$ the Haupt-Exponent of a (*i.e.* ξ is the minimum giving $a^\xi \equiv +1$), and the base (a) should be chosen so as to have a small Haupt-Exponent (ξ). This method is convenient only when ξ' or ξ'' is small, as otherwise the numerical labor is considerable.

Ex. Given $2^{460} \equiv +1$, and $2^{230} \equiv -1 \pmod{p=23041}$.
Here $2^{30} \equiv 8183$, and $2^{60} \equiv 4343 \pmod{p}$. Hence $y = \pm 4343$, and $y = \pm 8183$ are two roots of $y^4+1 \equiv 0$, $y^8+1 \equiv 0 \pmod{p}$, respectively.

11*c*. METHOD 3° (for $y^4+1 \equiv 0$). *From two known* 2^{10} *partitions.* If there be given *two* of

$$p \text{ or } p^\kappa = a^2+b^2 = c^2+2d^2 = e^2-2f^2 = 2f'^2-e'^2 \ldots\ldots (13),$$

or (by preference)

$$\left.\begin{array}{l} y_1^2+1 \equiv 0 \pmod{p \text{ or } p^\kappa}, \text{ together with } \textit{one} \text{ of} \\ p \text{ or } p^\kappa = c^2+2d^2 = e^2-2f^2 = 2f'^2-e'^2 \end{array}\right\} \ldots\ldots (13a),$$

then $(a\pm b)^4+4a^4 \equiv 0$, and $(a \pm b)^4+4b^4 \equiv 0 \pmod{p \text{ or } p^\kappa}$,

and $c^4 \equiv 4d^4$, $e^4 \equiv 4f^4$, $e'^4 \equiv 4f'^4 \ldots\ldots\ldots\ldots\ldots (\text{mod } p \text{ or } p^\kappa)$.

Hence, eliminating 4, the four roots of $(y^4+1)\equiv 0$ are given by

$$y \equiv \pm (y_1 \pm 1)y_2, \pmod{p \text{ or } p^\kappa} \quad \ldots\ldots\ldots\ldots (14),$$

wherein y_1 may be either root of $y_1^2+1\equiv 0 \pmod{p \text{ or } p^\kappa}$...(14*a*),

or $y_1 \equiv$ any one of $\frac{a}{b}, \frac{b}{a}; \frac{de}{cf}, \frac{cf}{de}; \frac{de'}{cf'}, \frac{cf'}{de'}, \pmod{p \text{ or } p^\kappa}$..(14*b*),

and $y_2 \equiv$ any one of $\frac{d}{c}, \frac{c}{2d}; \frac{f}{e}, \frac{e}{2f}, \frac{f'}{e'}, \frac{e'}{2f'}, \pmod{p \text{ or } p^\kappa}$...(14*c*),

The above gives a great choice of formulæ for the terms y_1, y_2 entering into y. This is a very convenient method when the denominators in y_1, y_2 are small or composed of small factors, as this renders the reduction of the fractions easy (Art. 9). The reduction of y_1 can be avoided if the roots (y_1) of $y_1^2+1\equiv 0$ are known; [see (14), (14*a*)].

Thus, by this Method the solution of $y^4+1\equiv 0 \pmod{p \text{ or } p^\kappa}$ is always possible when two (independent) partitions (13) can be* found.

Ex. Given $p=99961=275^2+156^2=293^2+2.84^2$.

$$\text{Here } y \equiv \frac{a-b}{b}\cdot\frac{c}{2d} \equiv \frac{275-156}{156}\cdot\frac{293}{2.84} = \frac{4981}{32.9.13} \pmod p$$

$$\equiv \frac{4981+6p}{32.9.13} = \frac{604747}{32.9.13} = \frac{46519}{32.9} \equiv \frac{-53442}{32.9} = \frac{-2969}{16} \equiv \frac{-2969+p}{16}$$

$$= \frac{96992}{16} = 6062 \pmod p.$$

Hence $y=\pm 6062$ are two roots of y^4+1.

Or, given $p=99961=293^2+2.84^2$, and $y_1=\pm 37804$ the roots of $y_1^2+1\equiv 0 \pmod p$.

Here $y\equiv(y_1-1)\cdot\frac{c}{2d}=37803.\frac{293}{2.84}=\frac{12601.293}{8.7}$, which also gives $y=6062$ on reduction.

11*d*. METHOD 4° (for $y^8+1\equiv 0$). *From a known 4-tan factorisation with certain 2^{16} partitions.* If there be given

$$N=(\alpha X^2)^4+(\beta Y^2)^4\equiv 0 \pmod{p \text{ or } p^\kappa, p=16\varpi+1;\ XY>1}..(15),$$

with one of $p=\alpha t^4 \pm \beta u^4$, or $=t'^4 \pm \alpha\beta u'^4$.....(15*a*),

* The author's Tables of *Quadratic Partitions*, London, 1904, give the (a, b), (c, d) partitions of all primes $p=8\varpi+1$ up to 100000: these enable $y^4+1\equiv 0 \pmod{p \text{ \& } p^4}$ to be *directly* solved to same limit.

or with one of p (or p^x) $= \alpha t_1^2 \pm u_1^2$,

and one of p (or p^x) $= t_2^2 \pm \beta u_2^2$...(15*b*),

or with 2^{ω} congruences to mod. p (or p^x) *of same form* as the 2^{ω} partitions;

then $\alpha^4 X^8 \equiv -\beta^4 Y^8 \pmod{p \text{ or } p^x}$.....(16*a*),

and $\beta^4 u^8 \equiv \alpha^4 t^8$, or $(\alpha\beta)^4 u'^8 \equiv t'^8$.....(16*b*),

or $\beta^4 u_2^8 \equiv t_2^8$, and $u_1^8 \equiv \alpha^4 t_1^8$.....(16*c*).

Eliminating α, β gives four roots (y) of $y^8 + 1 \equiv 0 \pmod{p \text{ or } p^x}$ in the form

$$y \equiv \pm \frac{Y}{X} \cdot y', \text{ or } \equiv \pm \frac{X}{Y} \cdot \frac{1}{y'} \pmod{p \text{ or } p^x} \text{.....(17)},$$

where $y' \equiv$ any of $\frac{t}{u}, \frac{t'}{\alpha u'}, \frac{\beta u'}{t'}, \frac{t_1 t_2}{u_1 u_2}$ .....(17*a*),

11*e*. *Simple Case* ($\alpha X^2 = 1$). Writing $\alpha X^2 = 1$ in (15), reduces it to the simpler form

$$N = 1 + (\beta Y^2)^4 \equiv 0 \pmod{p \text{ or } p^x} \text{.....(15')}.$$

This form is important, as it enables a *known* solution of $y^4 + 1 \equiv 0$ (wherein $y = \beta Y^2$) to be used instead of (15), as the starting datum. It suffices to write $\alpha = 1$, $X = 1$ in Results (15*a*), (17), (17*a*).

11*f*. *Simple Case* ($\alpha = 1$, $\beta = 2$). This is an important Case. Let there be given

$$N = X^8 + (2Y^2)^4 \equiv 0 \pmod{p \text{ or } p^x},\ p = 16\varpi + 1;\ XY > 1) \text{...(18)}.$$

with some of the 2^{ω} partitions, or congruences of same form

$$p \text{ or } p^x = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 = 2f'^2 - e'^2 \text{...(18*a*)},$$

Then $16Y^8 \equiv -X^8 \pmod{p \text{ or } p^x}$,

and $(a \pm b)^4 + 4a^4 \equiv 0$, $(a \pm b)^4 + 4b^4 \equiv 0$,

whence $(a \pm b)^8 = 16a^8$, and $(a \pm b)^8 = 16b^8$.....(19*a*),

or one of $c^8 \equiv 16d^8$, $e^8 \equiv 16f^8$, $e'^8 \equiv 16f'^8$.....(19*b*).

Eliminating 16 gives four roots of $y^8 + 1 \equiv 0 \pmod{p \text{ or } p^\kappa}$ in the form

$$y \equiv \pm \frac{Y}{X} \cdot y', \text{ or } \pm \frac{X}{Y} \cdot \frac{1}{y'} \pmod{p \text{ or } p^\kappa} \ldots\ldots (20),$$

where $y' \equiv$ any of $y_1 \pm 1$, or $y_2 \ldots\ldots\ldots (20a)$,

and

$$y_1 = \text{any of } \pm \frac{a}{b}, \pm \frac{b}{a}; \quad y_2 \text{ any of } \pm \frac{c}{d}, \pm \frac{e}{f}, \pm \frac{e'}{f'} \ldots (20b).$$

This gives a great choice of formulæ for the terms y_1, y_2 entering into y' and y.

Ex. (of Art. 11*d*).

Given $p = 54721 = 15^4 + 8^4 = 15^4 + 2^4.2^8$; here $\alpha = 15$, $\beta = 2$, $X = 1$, $Y = 2$,

Given also $p = 119^2 + 15.52^2 = 161^2 + 2.120^2$ (to eliminate α, β).

Then, by (17), (17*a*), $y = \pm Y \cdot \frac{t_1 t_2}{u_1 u_2} = \pm \frac{2.52.161}{119.120} = \pm \frac{13.23}{17.15} = \pm \frac{299}{17.15}$.

Reducing by Art. 9, $y \equiv \pm \frac{299 + 6p}{17.15} = \pm \frac{273904}{17.15} = \pm \frac{16112}{15} \pmod{p}$,

whence $y \equiv \pm \frac{16112 - 2p}{15} \equiv \mp \frac{93330}{15} = \mp 6222$, (two 8^{lo} roots).

Or, again, α, β may be eliminated by

$$p = 233^2 + 3.12^2 = 226^2 + 5.27^2 = 161^2 + 2.120^2.$$

Then, as by (17), (17*a*), $y = \pm Y \cdot \frac{t_1 t_2 t_3}{u_1 u_2 u_3} = \pm \frac{2.12.27.161}{233.226.120} = \pm \frac{27.7.23}{10.113.233}$, (or its reciprocal).

Hence $y \equiv \pm \frac{263290}{27.7.23}$, which yields on reduction (Art. 9) $y = \pm 18570$ (two 8^{lo} roots).

Ex. (of Art. 11*e*). Given $p = 64433 = 135^2 + 2.152^2$;

and $1 + 60^4 = 1 + 2^4.5^8 \equiv 0 \pmod{p}$.

By (20), (20*b*), $y = \pm \frac{1}{Y} \cdot \frac{1}{y'} = \pm \frac{1}{Y} \cdot \frac{d}{c} = \pm \frac{152}{5.135} = \pm \frac{152}{27.25}$.

Reducing by Art. 9, $y = \pm \frac{152 + 6p}{27.25} \equiv \pm \frac{386750}{27.25} \equiv \pm \frac{15470}{27} \pmod{p}$.

Also $y \equiv \pm \frac{15470 - 4p}{27} \equiv \mp \frac{242262}{27} = \mp \frac{26918}{3} \pmod{p}$,

$\equiv \mp \frac{26918 - p}{3} = \pm \frac{37515}{3} = \pm 12505$ (two 8^{lo} roots).

It will be evident now that the success of the general Method 4° (for finding roots of $y^8 + 1 \equiv 0$) requires that the auxiliary congruences (15*a*, *b*) or (18*a*) which are needed for eliminating α, β, should be either *given*, or else that they should be *easy to form*. The simple form (18) has the advantage that the auxiliary congruences (18*a*) are *always possible*, and that any *one* of them suffices: in this case moreover it is not really necessary to compute an *actual partition* (18*a*) of the modulus (p or p^κ) itself; for the (c, d), (e, f) partitions of the whole number N in (18) can be formed *algebraically* by Art. 4, and will suffice to yield the congruences (19*b*); the partitions of p have, however, the advantage of yielding smaller numbers (c, d), (e, f) than those of N, thus giving easier (subsequent) numerical work.

[Note that this Method 4° (for solving $y^8+1\equiv 0$) is not nearly so general as Method 3° (for solving $y^4+1\equiv 0$), as it is by no means easy to find suitable factorisable numbers (15), (15′), (18), together with the necessary auxiliary congruences (15*a*, *b*), (18*a*). In fact no general Method for solving $y^8+1\equiv 0$ appears to be known, except such as involve the solution of a 2^{kc} congruence (often a difficult matter)].

12. STEP ii. *Remaining roots.* When one root (y_1) of either congruence $y^4 + 1 \equiv 0$ or $y^8+1 \equiv 0$ (mod p or p^κ) is known then the complete set of four roots of the former, or eight roots of the latter (all $< p$ or p^κ), are given as the *least residues* of y_1^ω (ω *odd*), viz.

y_1, y_3, y_5, y_7 are Residues of y_1, y_1^3, y_1^5, y_1^7 for $y^4+1\equiv 0$...(21),

y_1, y_3, &c...y_{15} are Residues of $y_1, y_1^3, \ldots y_1^{15}$ for $y^8+1\equiv 0$...(22).

This suggests the following *systematic* mode of computing the roots.

Let $y_2 =$ least residue of y_1^2 (mod p or p^κ).....(23).

Then the roots may be found in succession, each from the preceding, by multiplying each root, as found, by y_2 and taking the *least residue* of the product,

$$y_3 \equiv y_2 \cdot y_1\ ;\ y_5 \equiv y_2 \cdot y_3\ ;\ y_7 \equiv y_2 \cdot y_5\ ;\ \text{and so on...(24).}$$

But, it suffices to compute up to *half the full number* of roots by the above Rule, *i.e.*

Only y_2 for $y^4 + 1 \equiv 0$; Only y_3, y_5, y_7 for $y^8 + 1 \equiv 0$...(25);

the remaining roots being given at once by simple subtraction

from the modulus (p or p^κ), since the set of roots can always be arranged in pairs (say y', y'') such that

$$y' + y'' = \text{the modulus } p \text{ or } p^\kappa \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

The roots can also be arranged in pairs whose products satisfy the reciprocal relations,

$$y_1 y_3 \equiv y_5 y_7 \equiv -1;\ y_1 y_7 \equiv y_3 y_5 \equiv +1;\ \text{for } y^4 + 1 \equiv 0 \ldots (27),$$

$$\left.\begin{aligned} y_1 y_7 \equiv y_3 y_5 \equiv y_9 y_{15} \equiv y_{11} y_{13} \equiv -1 \\ y_1 y_{15} \equiv y_3 y_{13} \equiv y_5 y_{11} \equiv y_7 y_9 \equiv +1 \end{aligned}\right\} \text{for } y^8 + 1 \equiv 0 \ldots\ldots (28),$$

in which the law of connexion of the subscripts is obvious.

13. *Tests of work.* When half the full number of roots has been obtained by the above systematic process one of the following Tests of the arithmetical accuracy of the work may be applied to the last root so obtained.

Roots of $y^4 + 1 \equiv 0$. Here y_3 would be the only root so computed.

$y_1 y_3$ should $\equiv -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29a)$,

$y_3 y_3 \equiv y_9$ should = one of the roots found by subtraction...$(29b)$,

y_3^2 should $\equiv -y_1^2$, or should be a root of $y^2 + 1 \equiv 0 \ldots\ldots (29c)$.

Roots of $y^8 + 1 \equiv 0$. Here y_7 would be the last root computed as above.

$y_1 y_7$ should $\equiv -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29d)$,

$y_3 y_7 \equiv y_9$ should = one of the roots found by subtraction...$(29e)$,

y_7^2 should $\equiv -y_3^2$, or should be a root of $y^4 + 1 \equiv 0 \ldots\ldots (29f)$,

Any of the above Tests will suffice.

14. *Previous Congruenee Tables.* Reuschle's *Tafeln complexer Primzahlen*, Berlin, 1875, gives—on pages 443, 446—short Tables of solutions of the two congruences $y^4 + 1 \equiv 0$, $y^8 + 1 \equiv 0 \pmod p$, extending only to $p \gtrsim 1000$. Only half the full number of roots is given in each case, viz. the roots $< \frac{1}{2}p$. On account of their small extent ($p \gtrsim 1000$), these Tables are of little use for factorisation of high numbers;

and—in consequence of the omission of one half of the roots—are not really convenient even for the search for small divisors (<1000).

15. *Use of Congruence-Tables in Factorisation.* Congruence-Tables, such as described above (Art. 8), give at once the prime divisors (p), and (to a lesser extent) the power-divisors (p^x), of all numbers $N=(Y^4+1)$, and (Y^8+1), where

$$Y=mp \pm y, \text{ or } =mp^x \pm y \ldots\ldots\ldots\ldots\ldots\ldots(30),$$

and $y=$ any root of $y^4+1\equiv 0$, or $y^8+1\equiv 0 \pmod{p \text{ or } p^x}$..(30*a*).

Such Congruence-Tables are therefore a *most powerful aid* to factorisation of such binomials giving the means (by a comparatively slight examination) of *complete factorisation* of *all* such numbers N and $\frac{1}{2}N$ up to the limits.

N, or $\frac{1}{2}N$, $<p_m^2$, where p_m is the prime next $>$ that for which the Congruence-Table is continuous,

and also of detection of *all* High Primes $P=N$ or $\frac{1}{2}N$ up to the same limit, and also of *complete factorisation* in certain cases up to much higher limits, viz. of all such numbers of form

$$N \text{ or } \tfrac{1}{2}N=(p_1 p_2 \ldots p_r)(p_a^{\alpha}.p_b^{\beta}\ldots).P\ldots\ldots\ldots(31),$$

where the primes (p_1, p_2, &c.) and prime-powers ($p_a^{\alpha}, p_b^{\beta}$, &c.) are among those whose roots (y) are given in the Congruence-Table, and P is a High Prime $<p_m^2$ (as above). The factorisation, and detection of High Primes, can be carried beyond those limits by special means.

[The author's Congruence-Table of $y^4+1\equiv 0$, being complete and continuous (Art. 8) up to $p_m=32441$ has served for the complete factorisation of $N=y^4+1$ up to $y=180$* and of $\frac{1}{2}N=\frac{1}{2}(y^4+1)$ up to $y=249$*, without a break; and also for many higher values of y; and also for the detection of all the High Primes* $\not> 10.10^7$ contained in $N=y^4+1$ up to $y=1000$; see Art. 6*b*, 7*a*].

High Factorisations, Ex.

$22946^4+1=41.673.10729.25913.36137$; [18 figures],

$92564^4+1=41.337.457.4057.42073.68113$; [20 figures].

* The factorisation was specially extended (as stated in Art. 6*b*) to $y=226$ in case of $N=(y^4+1)$, and to $y=265$ in case of $N=\frac{1}{2}(y^4+1)$, *continuously* (*i.e.* without break), by aid of MS. Tables of solutions of the Congruence $y^4+1\equiv 0$ (mod. p or p^4), compiled by the author, which are now continuous up to $p \not> 50000$. These Tables thus enabled the detection of High Primes up to 25.10^8.

16. *General Binomial Congruence* ($x > 1$). Solutions (y) of the more general congruences

$$a^4 + y^4 \equiv 0,\ a^8 + y^8 \equiv 0 \pmod{p \text{ or } p^\kappa};\ [\text{a constant}] \ldots (32),$$

could be found by various methods similar to those described above (Art. 8 to 13) for the special case when $a = 1$. Also known solutions (Y) of the simple case (where $a = 1$) may be utilised to yield solutions of the more general form by multiplying the simple congruences throughout by a^4 or a^8, whereby at once

$$y = \text{Least Residue of } aY \pmod{p \text{ or } p^\kappa} \ldots\ldots (32a).$$

When one root has been thus obtained, the other roots may be obtained in the same manner, or by the method of Art. 12.

In the case where $a = 2$, one half of the even roots are given (at sight) as the doubles of the smaller roots (Y) of the simple case when $a = 1$. For since two of the roots (say Y, Y') of $Y^4 + 1 \equiv 0$ and four of the roots (say Y, Y', Y'', Y''') of $Y^8 + 1 \equiv 0$ are always $< \frac{1}{2}p$ or $\frac{1}{2}p^\kappa$, hence

$y = 2Y, 2Y'$ are the even roots of $y^4 + 1 = 0 \ldots\ldots\ldots\ldots (33a)$,

$y = 2Y, 2Y', 2Y'', 2Y'''$ are the even roots of $y^8 + 1 \equiv 0 \ldots (33b)$,

and the remaining (odd) roots are given by subtraction as in (26).

17. *Residuacity and Modularity.* The short notation

$$(q/p)_e = +1, \text{ or } -1, \text{ denotes } q^{\frac{p-1}{e}} \equiv +1, \text{ or } -1 \pmod{p} \ldots (34),$$

$$\text{where } p = m.e + 1 = \text{prime}, \ [\text{here } e = 2^\kappa] \ldots\ldots\ldots (34a).$$

Here q is said to be a *Residue* or *Non-Residue* of order $e\,(= 2^\kappa)$ of the prime modulus (p); and conversely the prime p is said to be a *Modulus* or *Non-Modulus* of order e of the base q. These properties are styled the *Residuacity* of q, and *Modularity* of p. The question of whether $(q/p) = +1$, or -1, is in general completely determinable, when $e = 2, 4, 8$ (and also $e = 16$ where $q = \pm 2$) by the linear and quadratic relation of q to p, viz.

$$p = m.q + r = a^2 + b^2 = c^2 + 2d^2 = t^2 \pm qu^2 \ldots\ldots (35).$$

18. *Modularity of Quartans &c., Octavans, &c.* When the modulus is a Quartan, Half-Quartan. &c., *prime*, the rules

for 4^{ic}, 8^{ic}, and sometimes 16^{ic} modularity take simple forms for small bases q (especially for prime bases q). Thus

$$y = 2mq \text{ gives } (q/p)_8 = +1,$$
$$\text{for } p = (x^4 + y^4) \text{ and } (x^8 + y^8);\ [q \text{ prime}] \ldots (36),$$
$$x \mp y = 2mq \text{ or } 4mq \text{ gives } (q/p)_8 = +1,$$
$$\text{for } p = \tfrac{1}{2}(x^4 + y^4) \text{ and } \tfrac{1}{2}(x^8 + y^8) \ldots\ldots (37),$$

appear to be in general sufficient (though by no means always necessary) conditions for $(q/p)_8 = +1$, when q is an odd prime: when q is composite, the conditions are more complex. The detailed criteria of 2^{ic}, 4^{ic}, 8^{ic} residuacity of the small bases $\pm q = 2, 3, 5, \ldots, 12$ for such moduli are given in the four Tables A, B, C, D, following:

Criteria.	*Tab.*	*Mod.*	*Tab.*	*Mod.*
2^{ic}, 4^{ic}, 8^{ic},	A;	$p = x^4 + y^4$	B;	$p = \frac{1}{2}(x^4 + y^4)$
2^{ic}, 4^{ic}, 8^{ic}, 16^{ic},	C;	$p = x^8 + y^8$	D;	$p = \frac{1}{2}(x^8 + y^8)$

The same criterion applies to both $\pm q$ alike throughout these Tables, except in the two right-hand columns of Tab. B, D, wherein the criteria apply to only one of $\pm q$, viz. to that case which *gives the simplest criterion*, viz.

For $+q$ or $-q$, according as q, $\frac{1}{2}q$, $\frac{1}{4}q$, &c. $= (4k+1)$ or $(4k-1)$.

Tab. B; $p = \frac{1}{2}(x^4 + y^4) = 8\varpi + 1$; $(q/p)_8$ given for $q = +2, 5, 10; \bar{3}, \bar{6}, \bar{7}, \bar{11}, \bar{12}$;

Tab. D; $p = \frac{1}{2}(x^8 + y^8) = 16\varpi + 1$; $(q/p)_{16}$ given for $q = +2, 5, 10; \bar{3}, \bar{6}, \bar{7}, \bar{11}, \bar{12}$;

The cases of $\pm q$ are connected by the simple relations

$$p = \tfrac{1}{2}(x^4 + y^4) = 8\varpi + 1;\quad (q/p)_8 \cdot (\bar{q}/p)_8 = (-1)^{\varpi} \ldots\ldots\ldots (38a),$$
$$p = \tfrac{1}{2}(x^8 + y^8) = 16\varpi + 1;\quad (q/p)_{16} \cdot (\bar{q}/p)_{16} = (-1)^{\varpi} \ldots\ldots\ldots (38b).$$

These criteria have been reduced* by the author from the general criteria of 4^{ic} and 8^{ic} modularity for the small bases $(\pm q)$ stated.

[The signs * † ‡ § ¶ in Tables B, D denote *repetition* of the condition so marked in the column to left in which the sign first occurs. The sign, &c., indicates that an additional condition (not easily included in the Table) is needed. In the case of $q = \bar{11}$, the upper signs are to be used throughout, or the lower signs throughout, each line].

19. 16^{ic} *and* 32^{ic} *Modularity.* The criteria hitherto discovered extend only up to order 8 in general. In all cases

$$(q/p)_8 = +1 \text{ involves } (q/p)_{16} = \pm 1, \text{ [when } p = 16\varpi + 1] \ldots\ldots\ldots (39).$$

* And have also been tested on all quartan and half-quartan primes > 100000, (and in a few cases to a higher limit).

Criteria of $(q/p)_e = \pm 1$; $[e = 2, 4, 8]$. $\quad p = x^4 + y^4 = 16w + 1$; $[x = \omega,\ y = \varepsilon]$. TAB. A.

q	$(q/p)_2 = -1$	$(q/p)_2 = +1$	$(q/p)_4 = -1$	$(q/p)_4 = +1$	$(q/p)_8 = -1$	$(q/p)_8 = +1$
± 2	.	$y = \varepsilon$	$y = 2\omega$	$y = 2\varepsilon$	.	$y = 4i$
± 3	$xy \neq 3\varepsilon$	$xy = 3\varepsilon$	$x = 3\omega$	$y = 3\varepsilon$	.	$y = 6i$
± 5	$xy \neq 5\varepsilon$	$xy = 5\varepsilon$	$x = 5\omega$	$y = 5\varepsilon$	.	$y = 10i$
± 6	$xy \neq 6i$	$xy = 6i$	i. $y = 6\omega$ ii. $x = 3\omega,\ y = 4i$	$y = 6\varepsilon$	.	$y = 12i$
± 7	i. $xy \neq 7\varepsilon$ ii. $x \mp y \neq 7\omega$	$xy = 7\varepsilon$ $x \mp y = 7\omega$	.	$xy = 7\varepsilon$	$x = 7\omega$	$x = 3\omega,\ y = 2\omega$ $y = 7\varepsilon$
± 10	$xy \neq 10i$	$xy = 10i$	$x \mp y = 7\omega$.	. $y = 10\varepsilon$	. .	. $y = 20i$
± 11	i. $xy \neq 11\varepsilon$ ii. $2x \mp y \neq 11\omega$ iii. $x \mp 3y \neq 11\omega$	$xy = 11\varepsilon$ $3x \mp y = 11\omega$ $x \mp 3y = 11\omega$	i. $y = 10\omega$ ii. $x = 5\omega,\ y = 4i$ $x = 11\omega$ $3x \mp y = 11\omega$	$x = 5\omega,\ y = 2\omega$ $y = 11\varepsilon$. $x \mp 3y = 11\omega$	$x = 5\omega,\ y = 2\omega$ . . $x \mp 3y = 11\omega$	. $y = 11\varepsilon$ . .
± 12	$xy \neq 6i$	$xy = 6i$	$x = 3\omega$	$y = 6i$	$y = 6\omega$	$y = 12i$

Criteria of $(q/p)_e = \pm 1$; $[e = 2, 4, 8]$. $\quad p = \frac{1}{2}(x^4 + y^4) = 8w + 1$; $[x = \omega,\ y = \omega]$. TAB. B.

	$(q/p)_2 = -1$	$(q/p)_2 = +1$	$(q/p)_4 = -1$	$(q/p)_4 = +1$	q	$(q/p)_8 = -1$	$(q/p)_8 = +1$
± 2	.	$x = \omega,\ y = \omega$	$x \mp y = 4\omega$	$x \mp y = 4\varepsilon$	2	$x \mp y = 8\omega$	$x \mp y = 8\varepsilon$
± 3	$xy = 3\omega$	$x \mp y = 3\varepsilon$	.	$x \mp y = 3\varepsilon$	3	$x \mp y = 6\omega$	$x \mp y = 6\varepsilon$
± 5	$xy = 5\omega$	$xy = 5\omega$	$x \mp 2y = 5\omega$	$x \mp y = 5\varepsilon$	5	$x \mp y = 10\omega$	$x \mp y = 10\varepsilon$
± 6	$xy = 3i$	$x \mp y = 3\varepsilon$	i. $x \mp y = 12\omega$ & 2ω ii. $x \mp y = 4\omega$ & 6ω	$x \mp y = 12\varepsilon$ & 2ω $x \mp y = 4\varepsilon$ & 6ω	$\overline{6}$	i. $x \mp y = 24\omega$ & 2ω ii. $x \mp y = 8\varepsilon$ & 6ω	$x \mp y = 24\varepsilon$ & 2ω $x \mp y = 8\omega$ & 6ω
± 7	i. $xy \neq 7\omega$ ii. $x \mp y \neq 7\varepsilon$	$xy = 7\omega$ $x \mp y = 7\varepsilon$	$xy = 7\omega$.	. $x \mp y = 7\varepsilon$	$\overline{7}$	. .	. $x \mp y = 7\varepsilon$
± 10	$xy = 5i$	$x \mp y = 10i$ $*x \mp 2y = 5\omega$	i. $x \mp y = 20\omega$ & 2ω ii. $x \mp y = 4\omega$ & 10ω * & $x \mp y = 4\varepsilon$	$x \mp y = 20\varepsilon$ & 2ω $x \mp y = 4\varepsilon$ & 10ω * & $x \mp y = 4\omega$ †	10	$x \mp y = 40\omega$ & 2ω $x \mp y = 8\varepsilon$ & 10ω * & † &c.	$x \mp y = 40\varepsilon$ & 2ω $x \mp y = 8\omega$ & 10ω * & † &c.
± 11	i. $x \mp y \neq 11\varepsilon$ ii. $x \mp 2y \neq 11\omega$ iii. $2x \mp y \neq 11\omega$	$x \mp y = 11\varepsilon$ $x \mp 2y = 11\omega$ $2x \mp y = 11\omega$	. . .	† $x \mp y = 11\varepsilon$ ‡ $x \mp 2y = 11\omega$ § $2x \mp y = 11\omega$	$\overline{11}$	† & $xy = 4i \mp 1$ ‡ & $xy = 4i \pm 1$ § & $xy = 4i \pm 1$	† & $xy = 4i \pm 1$ ‡ & $xy = 4i \mp 1$ § & $xy = 4i \mp 1$
± 12	$xy = 3i$	$x \mp y = 6i$	.	$x \mp y = 6i$	$\overline{12}$	i. $x \mp y = 12\omega$ ii. $x \mp y = 4\varepsilon$ & 6ω	$x \mp y = 12\varepsilon$ $x \mp y = 4\omega$ & 6ω

Criteria of $(q/p)_e = \mp 1$; $[e = 2, 4, 8, 16]$. $p = x^8 + y^8 = 32\varpi + 1$, $[x = \omega, y = \varepsilon]$. TAB. C.

q	$(q/p)_2 = -1$	$(q/p)_2 = +1$	$(q/p)_4 = -1$	$(q/p)_4 = +1$	$(q/p)_8 = -1$	$(q/p)_8 = +1$	$(q/p)_{16} = -1$	$(q/p)_{16} = +1$
± 2	.	$y = \varepsilon$	.	$y = \varepsilon$	.	$y = \varepsilon$	.	$y = \varepsilon$
± 3	$xy \mp 3\varepsilon$	$xy = 3\varepsilon$	$x = 3\omega$	$y = 3\varepsilon$	.	$y = 3\varepsilon$	.	$y = 3\varepsilon$
± 5	$xy \mp 5\varepsilon$	$xy = 5\varepsilon$	$x = 5\omega$	$y = 5\varepsilon$	.	$y = 5\varepsilon$	.	$y = 5\varepsilon$
± 6	$xy \mp 6i$	$xy = 6i$	$x = 3\omega$, $y = \varepsilon$	$y = 6i$	.	$y = 6i$	.	$y = 6i$
± 7	$xy \mp 7\varepsilon$	$xy = 7\varepsilon$	.	$xy = 7\varepsilon$	$x = 7\omega$	$y = 7\varepsilon$	.	$y = 7\varepsilon$
± 7	$x \mp y \mp 7\omega$	$x \mp y = 7\omega$	$x \mp y = 7\omega$	.	.	.	.	.
± 10	$xy \mp 10i$	$xy = 10i$	$x = 5\omega$, $y = \varepsilon$	$y = 10i$	.	$y = 10i$	.	$y = 10i$
± 11	$xy \mp 11\varepsilon$	$xy = 11\varepsilon$	$x = 11\omega$	$y = 11\varepsilon$	.	$y = 11\varepsilon$	.	$y = 11\varepsilon$
± 11	$x \mp 2y \mp 11\omega$	$x \mp 2y = 11\omega$	$x \mp 2y = 11\omega$	.	.	.	.	.
± 11	$2x \mp y \mp 11\varepsilon$	$2x \mp y = 11\varepsilon$	.	$2x \mp y = 11\varepsilon$	$2x \mp y = 11\varepsilon$	.	.	.
± 12	$xy \mp 6i$	$xy = 6i$	$x = 3\omega$	$y = 6i$	.	$y = 6i$	.	$y = 6i$

Criteria of $(q/p)_e = \pm 1$; $[e = 2, 4, 8, 16]$. $p = \frac{1}{2}(x^8 + y^8) = 16\varpi + 1$; $[x = \omega, y = \omega]$. TAB. D.

q	$(q/p)_2 = -1$	$(q/p)_2 = +1$	$(q/p)_4 = -1$	$(q/p)_4 = +1$	$(q/p)_8 = -1$	$(q/p)_8 = +1$	q	$(q/p)_{16} = -1$	$(q/p)_{16} = +1$
± 2	.	$x = \omega$, $y = \omega$	.	$x = \omega$, $y = \omega$	$x \mp y = 4\omega$ & 2ω	$x \mp y = 4\varepsilon$	2	$x \mp y = 8\omega$	$x \mp y = 8\varepsilon$
± 3	$xy = 3\omega$	$x \mp y = 3\varepsilon$	.	$x \mp y = 3\varepsilon$	.	$x \mp y = 3\varepsilon$	3	$x \mp y = 6\omega$	$x \mp y = 6\varepsilon$
± 5	$xy = 5\omega$	$xy \mp 5\omega$	.	$xy \mp 5\omega$	$x \mp 2y = 5\omega$	$x \mp y = 5\varepsilon$	5	$x \mp y = 10\omega$	$x \mp y = 10\varepsilon$
± 6	$xy = 3\omega$	$x \mp y = 6i$	.	$x \mp y = 6i$ {	$x \mp y = 12\omega$ & 2ω	$x \mp y = 12\varepsilon$ & 2ω	6	i. $x \mp y = 24\omega$ & 2ω	$x \mp y = 24\varepsilon$ & 2ω
					$x \mp y = 4\omega$ & 6ω	$x \mp y = 4\varepsilon$ & 6ω		ii. $x \mp y = 8\varepsilon$ & 6ω	$x \mp y = 8\omega$ & 6ω
± 7	$xy \mp 7\omega$	$xy = 7\omega$	$xy = 7\omega$	.	.	.	7	i. .	.
± 7	$x \mp y \mp 7\varepsilon$	$x \mp y = 7\varepsilon$	.	$x \mp y = 7\varepsilon$	.	$x \mp y = 7\varepsilon$		ii. .	$x \mp y = 7\varepsilon$
± 10	$xy = 5\omega$ {	$x \mp y = 5\varepsilon$	.	{ $x \mp y = 20i$ & 2ω	$x \mp y = 20\omega$ & 2ω	$x \mp y = 20\varepsilon$ & 2ω	10	i. $x \mp y = 40\omega$ & 2ω	$x \mp y = 40\varepsilon$ & 2ω
				$x \mp y = 4i$ & 10ω	$x \mp y = 4\omega$ & 10ω	$x \mp y = 4\varepsilon$ & 10ω		ii. $x \mp y = 8\varepsilon$ & 10ω	$x \mp y = 8\omega$ & 10ω
		* $x \mp 2y = 5\omega$	.	$x \mp y = 4i$ & *	$x \mp y = 4\varepsilon$ & *	† $x \mp y = 4\omega$ & *		iii. *, & † &c.	* & † &c.
± 11	$x \mp y \mp 11\varepsilon$	$x \mp y = 11\varepsilon$	.	$x \mp y = 11\varepsilon$	.	‡ $x \mp y = 11\varepsilon$			
± 11	$x \mp 3y \mp 11\varepsilon$	$x \mp 3y = 11\varepsilon$	.	$x \mp 3y = 11\varepsilon$	.	§ $x \mp 3y = 11\varepsilon$	$\overline{11}$	‡ & $xy = 4i \mp 1$	‡ & $xy = 4i \pm 1$
± 11	$3x \mp y \mp 11\varepsilon$	$3x \mp y = 11\varepsilon$	.	$3x \mp y = 11\varepsilon$	.	¶ $3x \mp y = 11\varepsilon$		§ & $xy = 4i \pm 1$	§ & $xy = 4i \mp 1$
								¶ & $xy = 4i \pm 1$	¶ & $xy = 4i \mp 1$
± 12	$xy = 3i$	$x \mp y = 6i$	.	$x \mp y = 6i$	.	$x \mp y = 6i$	$\overline{12}$	i. $x \mp y = 12\omega$	$x \mp y = 12\varepsilon$
								ii. $x \mp y = 4\varepsilon$ & 6ω	$x \mp y = 4\omega$ & 6ω

No criteria are as yet known for $(q/p)_{16}$, except in the case when $q = \pm 2$, for which the criteria* are shown in the following Table, $[p = 16\varpi + 1$ throughout$]$:—

mod p ; $(2/p)_{16} = +1$	mod p ; $(2/p)_{16} = -1$; $(2/p)_{16} = +1$ .....
$(x^4 + y^4)$; $y = 2\varepsilon$	$\frac{1}{2}(x^4+y^4)$; $x \mp y = 16\omega$; $x \mp y = 16\varepsilon$....(40a);
$(x^8 + y^8)$; $y = \varepsilon$	$\frac{1}{2}(x^8+y^8)$; $x \mp y = 8\omega$; $x \mp y = 8\varepsilon$.....(40b).

These criteria are for $q = +2$; those for $q = \pm 2$ are connected by the relations (38a, b).

Also in all cases $(2/p)_{16} = +1$, involves $(2/p)_{32} = \mp 1$ (when $p = 32\varpi + 1$); but no criteria are known for the sign of $(2/p)_{32}$. Thus the known criteria for $(q/p)_\varepsilon = \mp 1$, where $e = 2^\varepsilon$, stop at $e = 16$ for the case of $y = \pm 2$, and at $e = 8$ when $q > 2$.

20. *New Criteria for* $(2/p)_{32}$ *and* $(q/p)_{16}$. It seems probable that the forms $p = (x^8 + y^8)$ and $\frac{1}{2}(x^8 + y^8)$ bear *much the same relations* to both $(2/p)_{32}$ and $(q/p)_{16}$ that $p = (x^4 + y^4)$ and $\frac{1}{2}(x^4 + y^4)$ are known to bear towards both $(2/p)_{16}$ and $(q/p)_8$. This seems to involve as criteria, in many cases sufficient, (but not always necessary)—

mod p	$(2/p)_{32} = -1$; $(2/p)_{32} = +1$	$(q/p)_{16} = -1$; $(q/p)_{16} = +1$ .....
$(x^8 + y^8)$	$y = 2\omega$; $= 2\varepsilon$	. ; $y = \varepsilon q$.....(41b),
$\frac{1}{2}(x^8 + y^8)$	$x \mp y = 16\omega$; $x \mp y = 16\varepsilon$	$x \mp y = 2\omega q$; $x \mp y = 2\varepsilon q$ (in some cases) } ...(41b),

or, more generally thus:—

if $\quad p = x^4 + y^4, \quad$ and $P = x^8 + y^8$, [same x, y],

$\quad p' = \frac{1}{2}(x^4 + y^4)$, and $P' = x^8 + y^8$, [same x, y],

then (when $q \not> 12$) the criteria of modularity of (p, P), (p', P') have—(with some slight modification found necessary by induction)—the *same form* in x, y in the following pairs

$$(2/p)_{16} = \mp 1 \ \& \ (2/P)_{32} = \mp 1; \ (2/p')_{16} = \mp 1 \ \& \ (2/P')_{32} = \mp 1 \text{..}(42a,b),$$

$$(q/p)_8 = \mp 1 \ \& \ (q/P)_{16} = \mp 1; \ (q/p')_8 = \mp 1 \ \& \ (q/P')_{16} = \mp 1 \text{..}(42c,d),$$

The two right-hand columns of Tables C, D preceding, have been drawn up in accordance with these assumed rules:

* These are due to the author: see his Paper *On 2 as a* 16^{th} *Residue* in *Proc. Lond. Math. Soc.*, Vol. XXVII, 1895, p. 85 *et. seq.*

these criteria are of course somewhat conjectural, as they depend only on formal analogy and on a small quantity of numerical evidence (given below).

20*a*. *Numerical evidence.* There are so few primes practically available of the forms $p=(x^8+y^8)$ and $\frac{1}{2}(x^8+y^8)$ that very little numerical testing can be tried. *All* the data at present) available are shown in the accompanying Table E. The body of the Table shews the residue-index ($e=2$, 4, 8, 16, 32, &c.) actually found (by computation) to give $(q/p)_e=-1$, or $+1$, as shown in the head-line: the column headed $(p-1)/\Omega$ shows the *highest power* of 2 contained in $(p-1)$.

TAB. E.

p	p	x, y	$\frac{p-1}{\Omega}$	$q=$ $(q/p)_e=$	2 $1, \overline{1}$	-3 $1, \overline{1}$	5 $1, \overline{1}$	-6 $1, \overline{1}$	-7 $1, \overline{1}$	10 $1, \overline{1}$	-11 $1, \overline{1}$	-12 $1, \overline{1}$
(x^8+y^8)	257	1, 2	256	$e\equiv$	16, 32	. 2	. 2	. 2	. 2	. 2	2, 4	. 2
	65537	1, 4	2^{16}	$e\equiv$	2^{15}, 2^{16}	. 2	. 2	. 2	. 2	. 2	. 2	. 2
	2070241	5, 6	32	$e\equiv$	16, 32	16, 32	2, 4	32, .	. 2	2, 4	. 2	16, 32
	100006561	3, 10	32	$e\equiv$	16, 32	2, 4	32, .	2, 4	2, 4	16, 32	. 2	2, 4
$\frac{1}{2}(x^8+y^8)$	198593	3, 5	64	$e\equiv$	8, 16	. 2	. 2	. 2	. 2	. 2	. 2	. 2
	21523361	1, 9	32	$e\equiv$	8, 16	. 2	. 2	. 2	. 2	16, 32	. 2	. 2
	1071182721	3, 11	128	$e\equiv$	8, 16	. 2	4, 8	. 2	32, 64	4, 8	. 2	. 2
	407865361	1, 13	16	$e\equiv$	4, 8	16, .	4, 8	4, 8	16, .	16, .	. 2	8, 16

21. *Residuacity of* 2 *to mod* (x^4+y^4), *and* (x^2+y^2), *&c.* It is worth noting that:—

If $p=x^4+y^4$ and $p'=x^2+y^2$ be *prime* (with same x, y),

then $(2/p)_{16}=+1$, $(2/p')_2=+1$ involve one another.....(43a).

Also,

if $p=\frac{1}{8}(x^4+y^4)$ and $p'=\frac{1}{2}(x^2+y^2)$ be *prime* (with same x, y),

then $(2/p)_{16}=+1$, $(2/p')_8=+1$ involve one another (if $p'=16\varpi'+1$)...(43b).

22. 32^{ic} *Residuacity of* 2 *with Quartans and Half-Quartans.* The 16^{ic} criteria of the base 2 with respect to Quartan and Half-Quartan primes are so extremely simple (Art. 19), that it seems probable that the 32^{ic} criteria *with such primes* should be much simpler than with primes in general, and therefore (in absence of any direct theory) more easily discoverable by numerical trial. As a step towards discovering such a criterion, the author has computed the actual (∓ 1) value* of $(2/p)_{32}$ for *all* primes p,

$p=x^4+y^4=32\varpi+1$, $\quad>9$ million, and some higher,

$p=\frac{1}{2}(x^4+y^4)=32\varpi+1$, $\quad>9$ million, and some higher.

The results are shown in Table VIII, which is divided (down the middle) into two parts:—

The left Table shows those primes for which $(2/p)_{32}=-1$,

The right Table shows those primes for which $(2/p)_{32}=+1$.

The column E shows the *highest power* of 2 in $(p-1)$, and the column e (on the right) shows the *highest power* of 2 in the residue-index, *i.e.* such that $(2/p)_e=+1$.

32. *Tables of primes* (Tab. I. to VII). These Tables, immediately following, are explained in Art. 7, 7*a*.

* The author has failed as yet in deducing any definite criterion from these results: but it seems worth while placing them on record (as they are the outcome of heavy work) for future use.

Quartan Primes, $p = (x^4 + y^4)$, [x *odd*, y *even*]. TAB. I.

p	x, y	p	x, y	p	x, y	p	x, y
17	1, 2	149057	17, 16	824641	11, 30	2070241	25, 36
97	3, 2	151057	19, 12	838561	13, 30	2085217	3, 38
257	1, 4	160001	1, 20	847601	25, 26	2168657	17, 38
337	3, 4	160081	3, 20	867281	29, 20	2279617	21, 38
641	5, 2	166561	9, 20	893521	17, 30	2351857	39, 14
881	5, 4	168737	19, 14	941537	29, 22	2378977	39, 16
1297	1, 6	204481	21, 10	944257	31, 12	2473441	39, 20
2417	7, 2	243521	17, 20	961937	31, 14	2522257	33, 34
2657	7, 4	260017	21, 16	988417	27, 26	2566561	9, 40
3697	7, 6	279857	23, 2	1049201	5, 32	2616577	27, 38
4177	3, 8	280097	23, 4	1050977	7, 32	2684161	37, 30
4721	5, 8	283937	23, 8	1055137	9, 32	2690321	19, 40
6577	9, 2	284881	15, 22	1089841	23, 30	2754481	21, 40
10657	9, 8	289841	23, 10	1146097	27, 28	2825777	41, 2
12401	7, 10	317777	17, 22	1178897	19, 32	2836961	35, 34
14657	11, 2	331777	1, 24	1224337	33, 14	2839841	23, 40
14897	11, 4	334177	7, 24	1328417	23, 32	2922737	37, 32
15937	11, 6	346417	11, 24	1336337	1, 34	2930737	41, 18
16561	9, 10	360337	13, 24	1336417	3, 34	3112321	5, 42
28817	13, 4	384817	23, 18	1336961	5, 34	3157537	41, 24
38561	13, 10	391921	25, 6	1338737	7, 34	3195217	17, 42
39041	5, 14	394721	25, 8	1342897	9, 34	3242017	19, 42
49297	13, 12	411361	25, 12	1345921	33, 20	3362017	39, 32
54721	15, 8	457057	3, 26	1350977	11, 34	3391537	23, 42
65537	1, 16	459377	7, 26	1364897	13, 34	3428801	43, 10
65617	3, 16	462097	19, 24	1466657	19, 34	3439537	43, 12
66161	5, 16	463537	9, 26	1501921	35, 6	3457217	43, 14
66977	13, 14	471617	11, 26	1521361	35, 12	3553777	37, 36
80177	11, 16	531457	27, 2	1682017	7, 36	3578801	43, 20
83537	17, 2	587297	19, 26	1763137	17, 36	3635761	41, 30
83777	17, 4	596977	27, 16	1800577	33, 28	3649777	39, 34
89041	15, 14	614657	1, 28	1809937	19, 36	3653057	43, 22
105601	5, 18	621217	9, 28	1874177	37, 2	3750577	43, 24
107377	7, 18	643217	13, 28	1874417	37, 4	3818977	29, 42
119617	11, 18	728017	29, 12	1878257	37, 8	3874337	41, 32
121937	17, 14	736817	23, 26	1912577	37, 14	3942577	21, 44
130337	19, 2	744977	19, 28	1959457	23, 36	3959297	37, 38
131617	19, 6	745697	29, 14	1972097	31, 32	4035217	31, 42
134417	19, 8	812257	29, 18	2034161	37, 20	4100641	45, 2
140321	19, 10	812401	7, 30	2043617	29, 34	4100881	45, 4

Tab. I. (*continued*).

Quartan Primes, $p = x^4 + y^4$, [x *odd*, y even].

p	x, y	p	x, y
4104721	45, 8	7435921	33, 50
4162097	41, 34	7439681	47, 40
4279537	27, 44	7506097	21, 52
4398577	39, 38	7591457	23, 52
4467377	43, 32	7813777	51, 32
4477457	1, 46	7843057	27, 52
4477537	3, 46	7891777	53, 6
4478081	5, 46	7894577	53, 8
4505377	41, 36	7900481	53, 10
4506017	13, 46	7911217	53, 12
4560977	17, 46	7928897	53, 14
4607777	19, 46	8050481	53, 20
4671937	21, 46	8124461	37, 50
4715281	45, 28	8222257	53, 24
4755137	43, 34	8324801	49, 40
4879937	47, 4	8503057	1, 54
4880977	47, 6	8503681	5, 54
4910897	41, 38	8505137	53, 28
4918097	47, 14	8531617	13, 54
5039681	47, 20	8586577	17, 54
5211457	47, 24	8627777	47, 44
5308417	1, 48	8633377	19, 54
5309041	5, 48	8812241	35, 52
5385761	41, 40	8939057	53, 32
5391937	17, 48		
5436961	45, 34		
5663377	33, 46		
5764817	49, 2		
5768897	49, 8		
5785537	49, 12		
5978801	43, 40		
6015697	29, 48		
6185761	45, 38		
6252401	7, 50		
6278561	13, 50		
6333521	17, 50		
6444481	21, 50		
6765217	51, 2		
6769297	51, 8		
6775201	51, 10		
6790897	39, 46		
6925201	51, 20		
6964817	47, 38		
6999457	51, 22		
7101137	49, 34		
7166897	43, 44		
7222177	51, 26		
7326257	11, 52		

Tab. III.

High Quartan Primes.

$p = (x^4 + y^4)$, [x *odd*, y *even*].

	p	x, y
C	9834497	1, 56
	29986577	1, 74
B	40960001	1, 80
	45212177	1, 82
	59969537	1, 88
B	65610001	1, 90
Da	100000081	3, 100
	100006561	9, 100
	126247697	1, 106
	193877777	1, 118
	303595777	1, 132
	384160001	1, 140
	406586897	1, 142
	562448657	1, 154
	655360001	1, 160
	723394817	1, 164
	916636177	1, 174
	1049760001	1, 180
	1416468497	1, 194
	1536953617	1, 198
	1731891457	1, 204
	1944810001	1, 210

Tab. IV.

High Half-Quartan Primes.

$p = \frac{1}{2}(1 + y)^4$, [y *odd*].

	p	x, y
B	12705841	1, 71
B	14199121	1, 73
BJ	21523361	1, 81
	56275441	1, 103
	60775313	1, 105
	81523681	1, 113
	87450313	1, 115
	100266961	1, 119
	138461441	1, 129
	273990641	1, 153
	370600313	1, 165
	407865361	1, 169
	427518041	1, 171
	784119601	1, 199
	849090841	1, 203
	883050313	1, 205
	1984563001	1, 251
	2249930281	1, 259

Half-Quartan Primes, $p = \frac{1}{2}(x^4 + y^4)$, [$x$ & y *odd*]. TAB. II.

p	x, y	p	x, y	p	x, y	p	x, y
1	1, 1	353641	29, 1	1975121	43, 27	4591801	49, 43
41	3, 1	353681	29, 3	2005841	41, 33	4617073	55, 17
313	5, 1	378953	29, 15	2057633	45, 11	4672553	55, 21
353	5, 3	405641	27, 23	2092073	45, 17	4715233	55, 23
1201	7, 1	450881	29, 21	2093801	39, 37	4795481	51, 41
3593	9, 5	461801	31, 3	2163193	41, 35	4928953	55, 29
4481	9, 7	462073	31, 5	2171161	43, 31	4932713	49, 45
7321	11, 1	465041	31, 9	2190233	45, 23	5101961	53, 39
8521	11, 7	471041	31, 13	2439881	47, 3	5278001	57, 1
10601	11, 9	487073	31, 15	2440153	47, 5	5319761	57, 17
14281	13, 1	548953	29, 25	2441041	47, 7	5473313	57, 25
14321	13, 3	559001	31, 21	2447161	47, 11	5654641	53, 43
14593	13, 5	593273	33, 5	2454121	47, 13	5822441	51, 47
21601	13, 11	594161	33, 7	2481601	47, 17	5988193	55, 41
26513	15, 7	750313	35, 1	2537081	47, 21	6028313	57, 35
32633	15, 11	750353	35, 3	2705561	47, 27	6058993	59, 5
41761	17, 1	757633	35, 11	2793481	47, 29	6083993	59, 15
41801	17, 3	764593	35, 13	2866121	43, 39	6123841	59, 19
42073	17, 5	792073	35, 17	2882441	49, 3	6198601	59, 23
42961	17, 7	815401	31, 29	2901601	47, 31	6253993	59, 25
49081	17, 11	937121	37, 3	2907713	49, 15	6265001	51, 49
56041	17, 13	940361	37, 9	2947561	49, 19	6324401	59, 27
66361	19, 7	951361	37, 13	3032801	47, 33	6412321	59, 29
67073	17, 15	1002241	37, 19	3122281	43, 41	6520441	59, 31
72481	19, 11	1016033	35, 27	3148121	49, 27	6690881	57, 41
90473	19, 15	1054721	33, 31	3190153	47, 35	6922921	61, 1
97241	21, 1	1132393	37, 25	3236041	49, 29	6930241	61, 11
97553	21, 5	1156721	39, 1	3344161	49, 31	6948233	61, 15
104561	21, 11	1157033	39, 5	3383801	51, 7	6995761	59, 37
106921	19, 17	1198481	39, 17	3522521	51, 23	7020161	61, 21
111521	21, 13	1398841	37, 31	3577913	51, 25	7215401	59, 39
139921	23, 1	1414081	41, 7	3736241	51, 29	7471561	59, 41
141121	23, 7	1416161	41, 9	3759713	45, 43	7768081	59, 43
165233	23, 15	1420201	41, 11	3819481	49, 37	7941641	63, 19
195353	25, 3	1510121	41, 21	3948521	53, 9	8160401	57, 49
198593	25, 9	1510361	39, 29	3952561	53, 11	8230121	63, 29
205081	23, 19	1618481	39, 31	3987001	53, 17	8338241	63, 31
237073	25, 17	1678601	41, 27	4132913	51, 35	8925313	65, 1
237161	23, 21	1687393	37, 35	4295281	49, 41	8928593	65, 9
266921	27, 7	1709713	43, 5	4298881	53, 29	8967073	65, 17
280001	27, 13	1710601	43, 7	4319681	51, 37		
307481	27, 17	1734713	43, 15	4589593	55, 13		

TAB. V.

High Primes $p=\frac{1}{\mu}.(1+y^4)$, [y *even*]; $\mu=17, 41, 73, 89, 97,$ &c.

p	y,	μ	p	y,	μ	p	y,	μ
9167489	712	μ	36268129	354	μ	165991393	468	μ
9793969	380	μ	36269593	950	μ	194213177	564	μ
10088489	934	μ	39818929	392	μ	201796057	626	μ
10677089	3266	μ	40054897	356	μ	205048201	904	μ
11165137	528	μ	40514561	162	17	206063593	368	89
11505017	942	μ	44669593	736	μ	210378169	14506	μ
11966641	252	μ	44711201	548	μ	210469913	14506	μ
12321041	474	μ	49916473	378	μ	238275601	994	μ
13294121	502	μ	50855561	486	μ	241632361	740	μ
13374089	336	μ	50897897	330	μ	273148633	890	μ
14394409	362	μ	51244313	572	μ	287803777	834	μ
14579681	970	μ	51483121	172	17	L 308761441	8192	μ
14641849	456	μ	52216841	608	μ	334140193	762	μ
15120673	766	μ	57734881	676	μ	361562353	280	17
L 15790321	128	17	60880681	872	μ	389961553	830	μ
16673401	674	μ	63798737	668	μ	400495049	802	μ
16782449	326	μ	65798849	700	μ	431830177	470	μ
16898729	684	μ	73805233	680	μ	463504289	636	μ
17137129	514	μ	73853993	644	μ	463891201	298	17
17957969	956	μ	74046641	666	μ	535609489	496	μ
18145313	388	μ	74524553	728	μ	563676649	602	μ
18468497	434	μ	75297473	724	μ	599786777	396	41
19050289	750	μ	77938409	586	μ	606454393	482	89
20160553	402	μ	78374441	816	μ	611416873	870	μ
20361377	458	μ	82509577	814	μ	613350137	460	73
20905193	726	μ	86631049	282	73	613775969	718	μ
21333761	138	17	94106561	422	μ	670464121	744	μ
L 22253377	4096	μ	97089257	686	μ	707646281	558	μ
22925033	806	μ	97905289	630	μ	714666481	332	17
24132457	416	μ	98672257	446	μ	775275233	688	μ
24290249	398	μ	Lf 99990001	1000	μ	788278297	424	41
25068521	960	μ	113607841	324	97	825799841	532	97
25397761	770	μ	113947529	302	73	937534777	836	
25737017	464	μ	118821361	212	17			
25744921	366	μ	126041329	908	μ			
27126929	862	μ	126431801	976	μ			
27475081	372	μ	134472673	444	μ			
29497513	544	μ	137123009	534	μ			
31142473	756	μ	141456017	722	μ			
31582673	592	μ	157341673	344	89			

TAB. VI.

High Primes, $p=\frac{1}{\mu}\cdot\frac{1}{2}(1+y^4)$, [$y$ *odd*]; $\mu=17, 41, 73, 89, 97,$ &c.

p	y, μ	p	y, μ	p	y, μ
9037817	959 μ	31308961	779 μ	186643993	825 μ
9085337	283 μ	33510401	295 μ	210907993	291 17
9226673	699 μ	33621673	813 μ	219192097	737 μ
9485321	167 41	34040569	279 89	233726369	937 μ
9661777	547 μ	37529113	189 17	240110729	887 μ
9946609	627 μ	39006769	767 μ	268083401	727 μ
10316017	197 73	39785017	369 μ	288959497	967 μ
10509841	303 μ	40124537	755 μ	319585921	651 μ
10771417	347 μ	41912953	877 μ	334629161	407 41
11616697	433 μ	B 42521761	243 41	436337753	349 17
11756681	831 μ	42526489	195 17	468260633	549 97
12004217	215 89	43026433	585 μ	468571633	899 μ
12452641	797 μ	43068329	495 μ	478014457	687 μ
12602857	915 μ	45509137	697 μ	492387713	759 μ
12732529	327 μ	45721937	663 μ	499445449	733 μ
13001489	145 17	50088697	695 μ	504988801	897 μ
13068697	209 73	51909329	493 μ	508142377	751 μ
13974721	621 μ	52048313	519 μ	522026489	365 17
14042233	907 μ	52333297	377 μ	552784657	533 73
14160017	7453 μ	53152753	709 μ	563102449	785 μ
14414377	457 μ	53203889	955 μ	632133361	3125 μ
14751089	983 μ	58175849	319 89	635151689	849 μ
14896841	631 μ	60539593	213 17	701849009	775 μ
15290753	151 17	60665273	867 μ	793707041	985 μ
15499417	599 μ	65886001	943 μ	889334833	417 17
15601081	901 μ	66062657	793 μ	1094286241	9999 μ
16230041	191 41	68530937	469 μ		
17137793	385 μ	69593033	903 μ		
17522137	483 μ	87748937	429 μ		
17808841	921 μ	93621401	851 μ		
19007873	439 μ	95392169	541 μ		
20253553	391 μ	100104161	301 41		
23019641	511 μ	108003089	873 μ		
23754217	255 89	116490961	885 μ		
24840737	535 μ	119577209	953 μ		
24953633	633 μ	128307953	257 17		
27093617	803 μ	131579017	581 μ		
27760481	975 μ	135447881	375 73		
28271569	425 μ	144553441	2121 μ		
D 29423041	625 μ	183377633	281 17		

TAB. VII.

Octavan & Half-Octavan Primes.

	$p=x^8+y^8$	$x,\ y$		$p=\frac{1}{2}(x^8+y^8)$	$x,\ y$
$p<9.10^6$	257	1, 2	$p<9.10^6$	1	1, 1
	65537	1, 4		198593	3, 5
	2070241	5, 6			
$p>9.10^6$	100006561	3, 10	$p>9.10^6$	BJ 21523361	1, 9
	[None with $y \not> 38$]	1, y		107182721	3, 11
				407865361	1, 13

High Prime Factors (p) of Octavans.

[$\mu=17, 41, 73, 89, 97$, &c.]

$p=\frac{1}{\mu}(x^8+y^8)$		$y,\ \mu$	$p=\frac{1}{\mu}\cdot\frac{1}{2}(x^8+y^8)$		$y,\ \mu$
L	22253377	64, μ	D	22191649	35, μ
	37642417	34, μ		29423041	25, μ
	57734881	26, μ		45534289	39, μ
	113607841	18, 97		70978049	79, μ
	164819521	122, μ		262965473	77, μ
	221201713	112, μ		291295393	71, μ
	291444977	54, μ		454677073	61, μ
	396622273	108, μ		502761569	155, μ
L	4278255361	32, μ			

TAB. VIII.

32^{le} *Residuacity of* 2 *with Quartan and Half-Quartan Primes.*

	$(2/p)_{32} = -1$				$(2/p)_{32} = +1$				
p	p	E	x,	y	p	E	x,	y	e
$p = x^4 + y^4$	257	256	1,	4	10657	32	9,	8	32
	2657	32	7,	4	65537	65536	1,	16	2^{11}
	54721	64	15,	8	83777	64	17,	4	64
	149057	64	17,	16	160001	256	1,	20	32
	166561	32	9,	20	243521	64	17,	20	64
	280097	32	23,	4	283937	32	23,	8	32
	334177	32	7,	24	331777	4096	1,	24	64
	614657	256	1,	28	394721	32	25,	8	32
	944257	128	31,	12	411361	32	25,	12	32
	1050977	32	7,	32	621217	32	9,	28	32
	1328417	32	23,	32	1055137	32	9,	32	32
	1682017	32	7,	36	1345921	128	33,	20	32
	1972097	128	31,	32	1763137	64	17,	36	64
	2070241	32	25,	36	1800577	128	33,	28	32
	3157537	32	41,	24	1959457	32	23,	36	32
	3874337	32	41,	32	2378977	32	39,	16	32
	4505377	32	41,	36	2473441	32	39,	20	32
	5039681	64	47,	20	2566561	32	9,	40	32
	5308417	65536	1,	48	2839841	32	23,	40	32
	5385761	32	41,	40	3362017	32	39,	32	32
	5785537	64	49,	12	4879937	64	47,	4	64
	7439681	64	47,	40	5211457	64	47,	24	64
	8324801	64	49,	40	5391937	64	17,	48	64
	8627777	64	47,	44	5768897	64	49,	8	32
	9834497	4096	1,	56	7591457	32	23,	52	32
	303595777	256	1,	132	40960001	65536	1,	80	512
	384160001	256	1,	140	59969537	4096	1,	88	32
					655360001	2^{20}	1,	160	32
$p = \frac{1}{2}(x^4 + y^4)$	67073	512	17,	15	1054721	2048	33,	31	64
	1416161	32	41,	9	2907713	64	49,	15	64
	2481601	64	47,	17	5473313	32	57,	25	32
	4715233	32	55,	23	5988193	32	55,	41	32
	8925313	128	65,	1	8388241	64	63,	31	32
					13846144 1	256	129,	1	32